THE GEOMETRY OF SOLVABILITY AND DUALITY IN LINEAR PROGRAMMING

BY

ADI BEN-ISRAEL*

ABSTRACT

Solvability and boundedness criteria for dual linear programming problems are given in terms of the problem data and the intersections of the nonnegative orthant with certain complementary orthogonal subspaces.

Introduction. The duality theorem of linear programming(1) relates two linear extremum problems in terms of solvability, boundedness and equality of functional values. The classical theory of Lagrange multipliers admits extensions to some special nonlinear cases(2) as well as interpretations of duality in the context of applications(3).

Tucker, in [16], showed duality—in the linear case—to follow from elementary geometric considerations of complementary orthogonality of manifolds corresponding to the dual problems(4).

In this paper we follow Tucker's approach and supplement his results [16] by an alternative theorem for dual programs (Theorem 4 below), and by a characterization of all duality situations in terms of the geometrical configurations of certain manifolds associated with the data and the data itself (Theorem 6 below).

None of our results seem to be essentially new; yet our efforts may be justified for pedagogical reasons.

NOTATIONS. In this note we use the same notations as in [1]. In particular:
$\mathscr{F}$ is an *arbitrary ordered field*;
E^n is the n-dimensional vector space over $\mathscr{F}$;
$C\{f_1,\cdots,f_k\}$ is the *cone* spanned by the vectors $f_1,\cdots,f_k$ in E^n;

Received August 22, 1963

*This research was partly supported by the Office of Naval Research, contract Nonr-1228(10), project NR 047–021, and by the National Science Foundation, project G-14102.

(1) Conjectured by Von Neumann (e.g. [7], p. 23) and proved by Gale, Kuhn and Tucker [8]. For the extended form discussed here, see Charnes and Cooper [3].

(2) Notably Kuhn and Tucker [11].

(3) E.g. [7], pp. 19–22.

(4) A similar approach was used in [1] to develop, in a unified manner, the main theorems of linear inequalities.

A is an $m \times n$ matrix over $\mathscr{F}$;

$N(A)$ is the *null space* of A in E^n;

$R(A^T)$ is the *range space* of A^T in E^n.

In addition, let A^+ denote the generalized inverse of A (see, e.g., [14] and [2]).

The following two theorems, which are Corollaries 3′ and 5 respectively of [1], are used in the sequel and quoted here for ease of reference:

0.1 THEOREM. *Let L, $L^\perp$ be complementary orthogonal subspaces in E^n. Then the following are equivalent*:

(i) $L \cap E^n_+ = \{\theta\}$;

(ii) $L^\perp \cap \operatorname{int} E^n_+ \neq \phi$.

0.2 THEOREM. *Let L be a subspace of E^n of dimension k, $k = 1,2,\cdots,n$. Then the following are equivalent*:

(i) $L \cap \operatorname{bd} E^n_+ = C\{e_1,\cdots,e_p\}$, $1 \leqq p \leqq k$;

(ii) *$L^\perp$ has a basis in* $\operatorname{int} C\{e_{p+1},\cdots,e_n\}$.

1. LEMMA. *Let L be an arbitrary subspace of E^n. Then the following are equivalent*:

(i) $\{x + L\} \cap E^n_+ \neq \phi$ *for all* $x \in E^n$;

(ii) $L \cap \operatorname{int} E^n_+ \neq \phi$;

(iii) $\{x + L\} \cap \operatorname{int} E^n_+ \neq \phi$ *for all* $x \in E^n$.

Proof.

(i) $\Rightarrow$ (ii)

Suppose (ii) is false. This is possibly only in two cases:

Case A: $L \cap \operatorname{bd} E^n_+ = C\{e_1,\cdots,e_p\}$, $1 \leqq p \leqq n\text{-}\dim L$.

Case B: $L \cap E^n_+ = \{\theta\}$.

We will now show (i) to be false by producing, in each case, a vector x_0 such that $\{x_0 + L\} \cap E^n_+ = \phi$.

Case A: Let x_0 be any vector in $L^\perp \cap \operatorname{int} C\{-e_{p+1},\cdots,-e_n\}$, a set which is nonempty by Theorem 0.2.

Case B: Let x_0 be any vector in $L^\perp \cap \operatorname{int}\{-E^n_+\}$; the latter set is nonempty by Theorem 0.1.

(ii) $\Rightarrow$ (iii)

Let $y \in L \cap \operatorname{int} E^n_+$. Then for all $x \in E^n$ and scalars λ satisfying

$$\lambda > \max_{x_i < 0} \frac{|x_i|}{y_i}$$

we have $$x + \lambda y \in \operatorname{int} E^n_+.$$

(iii) $\Rightarrow$ (i) Obvious.

2. COROLLARY. *Let A be an arbitrary $m \times n$ matrix over $\mathscr{F}$. Then the following are equivalent*:

(i) $Ax = b, \quad x \geqq \theta$ *is solvable for all* $b \in R(A)$;
(ii) $Ax = \theta, \quad x > \theta$ *is solvable*;
(iii) $Ax = b, \quad x > \theta$ *is solvable for all* $b \in R(A)$.

Proof.
The solutions of $Ax = b$, when solvable, form the manifold $A^+ b + N(A)$, where A^+ denotes the generalized inverse of A, [14]. Now (i), (ii) and (iii) are the corresponding parts in Lemma 1 with $x = A^+ b$ and $L = N(A)$.

3. Corollary. *Let A be an arbitrary $m \times n$ matrix over $\mathscr{F}$. Then the following are equivalent*:
(i) $A^T w \geqq c$ *is solvable for all* $c \in E^n$;
(ii) $A^T w > \theta$ *is solvable*;
(iii) $A^T w > c$ *is solvable for all* $c \in E^n$.

Proof. In Lemma 1 let $L = R(A^T)$, $x = -c$.

4. Theorem. *Let A be an arbitrary $m \times n$ matrix over $\mathscr{F}$. Consider the system of equations and inequalities*:

I) $Ax = b,\ x \geqq \theta$	I′) $Ax = \theta,\ x \geqq \theta$
II) $A^T w \geqq c$	II′) $A^T w \geqq \theta$

Then:
a) I *is solvable for all* $b \in R(A)$ *if and only if* II′ *does not have solutions with nonnegative nonzero vectors* $A^T w$.
b) II *is solvable for all* $c \in E^n$ *if and only if* I′ *does not have nonnegative nonzero solutions.*
c) *If* II′ *has solutions* w *with* $A^T w \geq \theta$ *then* I *is solvable if and only if* $A^T w \geq \theta \Rightarrow (b, w) \geqq 0$.
d) *If* I′ *has solutions* x *with* $x \geq \theta$ *then* II *is solvable if and only if* $Ax = \theta,\ x \geq \theta \Rightarrow (c, x) \leqq 0$.

Proof.
a) By Corollary 2 it follows that I is solvable for all $b \in R(A)$ if and only if $N(A) \cap \operatorname{int} E^n_+ \neq \phi$. By Theorem 0.1, the latter condition is equivalent to $R(A^T) \cap E^n_+ = \{\theta\}$.
b) By Corollary 3, II is solvable for all $c \in E^n$ if and only if $R(A^T) \cap \operatorname{int} E^T_+ \neq \phi$. This is equivalent, by Theorem 0.1 to: $N(A) \cap E^n_+ = \{\theta\}$.
c) This is the well-known Farkas' lemma (see, e.g. [15]).
d) This is Theorem 1 in Ky Fan [6].

Remarks.
a) Theorem 4 is a collection of classical results in a setup which is completely analogous to "Fredholm's Alternative" theorem for linear equations [5]. Solvability relations between linear inequalities and equations were studied by

Motzkin [13], Kuhn [10] and generalized by Ky Fan to the case of complex normed linear spaces [6]. For a use of Fredholm's theorem to prove the main theorems of linear inequalities see [1].

b) For $b \in R(A)$, part c can be rewritten as:

c') If $b \in R(A)$ and II' has solutions w with $A^T w \geq \theta$ then I is solvable if and only if

$$A^T w \geq \theta \Rightarrow (A^T w, A^+ b) \geqq 0.$$

This follows from the fact that AA^+ is the perpendicular projection on $R(A)$, e.g. [2].

5. Let A be an arbitrary $m \times n$ matrix over $\mathscr{F}$, $b \in E^m$ and $c \in E^n$. Let

$$S = \{x \in E^n : Ax = b,\ x \geqq \theta\} \qquad I_1 = \sup_{x \in S}(c, x)$$

$$T = \{w \in E^m : A^T w \geqq c\} \qquad I_2 = \inf_{K \in T}(b, w)$$

The duality theorem of linear programming relates the problem of solving for I_1 the *primal* problem, to that of solving for I_2, the *dual* problem.

The duality theorem states indeed that there are four mutually exclusive cases:

Case A: $S \neq \phi$, $T \neq \phi$, $I_1 = I_2$

Case B: $S = \phi$, $T \neq \phi$, $I_2 = -\infty$

Case C: $S \neq \phi$, $T = \phi$, $I_1 = \infty$

Case D: $S = \phi$, $T = \phi$

Conjectured by von Neumann (see [7], p. 23) and proved by Gale, Kuhn and Tucker [8], this theorem was extended to some nonlinear situations, the most general being that of Charnes, Cooper and Kortanek [4].

We will now elaborate on the four cases given above. In terms of the data $\{A, b, c\}$, and more specifically of the configurations of $N(A)$ and $R(A^T)$ with respect to E^n_+, we give below conditions for the attainment of each of the above cases.

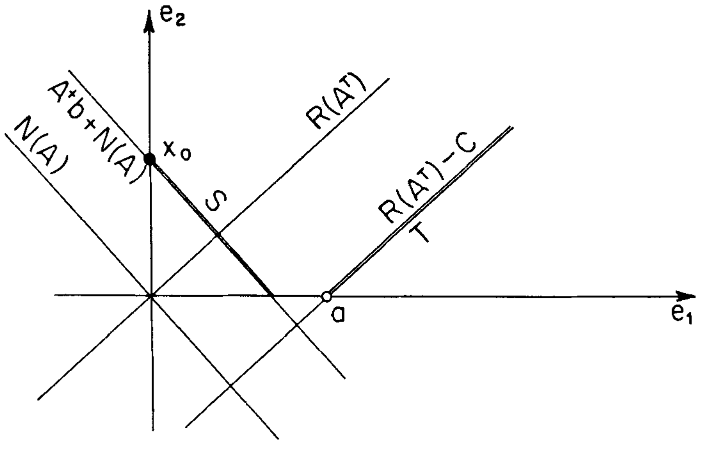
6. Theorem. *Let A be an arbitrary m n matrix over $\mathscr{F}$, $b \in R(A)$ in E^m, $c \in E^n$ and let S, T, I_1 and I_2 be as above. Then there are eight mutually exclusive cases, tabulated below.*

Proof. The cases $1, \cdots, 8$ are clearly mutually exclusive. In each case Theorem 4 is used to draw the conclusions regarding the sets S and T. Then the duality theorem of linear programming is used to obtain I_1 and I_2.

Remarks.

a) The above 8 cases can be visualized geometrically in a manner which helps to clarify the concept of duality. Thus in the 2-dimensional case where A is a 1×2 matrix, $\dim R(A^T) = \dim N(A) = 1$, the first case appears as follows:

Case	Assumptions: Intersection with E^n_+	Assumptions: Conditions on b, c	Conclusions: S	Conclusions: T	Conclusions: I_1, I_2
1	$N(A) \cap E^n_+ = \{\theta\}$	$A^T w \geq \theta \Rightarrow (A^T w, A^+ b) \geqq 0$	Non-empty	Non-empty for all $c \in E^n$	$I_1 = I_2$
2		$A^T w \geq \theta$ and $(A^T_m w, A^+ b) <$ for some w	Empty	,,	$I_2 = \infty$
3	$R(A^T) \cap E^n_+ = \{\theta\}$	$Ax = \theta,\ x \geq \theta \Rightarrow (c, x) \leqq 0$	Non-empty for all $b \in R(A)$	Non-empty	$I_1 = I_2$
4		$Ax = \theta,\ x \geq \theta$ and $(c, x) > 0$ for some x	,,	Empty	$I_1 = \infty$
5	$N(A) \cap \operatorname{bd} E^n_+ = C\{e_1 \cdots e_p\}$ $1 \leqq p \leqq \dim N(A)$ $R(A^T) \cap \operatorname{int} C\{e_{p+1}, \cdots, e_n\} \neq \phi$	$A^T w \geq 0 \Rightarrow (A^T w, A^+ b) \geqq 0$ and $Ax = \theta,\ x \geq \theta \Rightarrow (c, x) \leqq 0$	Non-empty	Non-empty	$I_1 = I_2$
6		$A^T w \geq \theta \Rightarrow (A^T w, A^+ b) \geqq 0$ but $Ax = \theta$, and $(c, x) > 0$ for some $x \geq 0$	,,	Empty	$I_1 = \infty$
7		$A^T w \geq \theta$ and $(A^T w, A^+ b) < 0$ for some w but $Ax = \theta,\ x \geq \theta \Rightarrow (c, x) \leqq 0$	Empty	Non-empty	$I_2 = -\infty$
8		$A^T w \geq \theta$ and $(A^T w, A^+ b) < 0$ for some w and $Ax = \theta,\ x \geq \theta$ and $(c, x) > 0$ for some x	,,	Empty	



The other cases are drawn in a similar manner. Furthermore, by the "complementary slackness" property, it is now easy to identify optimal points. Thus x_0 is the optimal solution of the primal problem and $a = A^T w_0 - c$, where w_0 is the optimal solution of the dual problem ([16], p. 15).

b) Theorem 6 combines well-known solvability theorems (Tucker [16], Charnes-Cooper [3], p. 214), and the duality theorem of linear programming to characterize the duality situations in terms of the data $\{A, b, c\}$.

Acknowledgment. The help and encouragement of Professor A. Charnes are hereby gratefully acknowledged.

References

1. Ben-Israel, A., Notes on linear inequalities, I: The intersection of the nonnegative orthant with complementary orthogonal subspaces, ONR Research Memorandum, No. 78; (to appear in *J. Math. Anal. Appl.*).
2. Ben-Israel, A. and Charnes, A., 1963, Contributions to the theory of generalized inverses, *J. Soc. Ind. Appl. Math.* (to appear).
3. Charnes, A. and Cooper W. W., 1961, *Management Models and Industrial Applications of Linear Programming*, Vols. I, II, J. Wiley and Sons, New York.
4. Charnes, A., Cooper ,W. W. and Kortanek, K., 1962, *Proc. Nat. Acad. Sci. U.S.A.*, **48**, 783–786.
5. Courant, R. and Hilbert, D., 1953, *Methods of Mathematical Physics*, Vol. I, Interscience Publishers, Inc., New York.
6. Fan, Ky., pp. 99–156 in [12].
7. Gale D., 1960, *The Theory of Linear Economic Models*, McGraw-Hill Book Co., New York.
8. Gale, D., Kuhn, H. W. and Tucker, A. W., pp. 317–319 in [9].
9. Koopmans, T. C. (editor), 1951, *Activity Analysis of Production and Allocation*, Cowles Monograph No. 13, J. Wiley and Sons, New York.
10. Kuhn, H. W., 1959, *Amer. Math. Monthly*, **63**, 217–232.

11. Kuhn, H. W. and Tucker, A. W., 1951, Nonlinear Programming, *Proc. Second Berkeley Symposium on Mathematical Statistics and Probability*, pp. 481–492.

12. Kuhn, H. W. and Tucker, A. W. (editors), 1956, *Linear Inequalities and Related Systems*, Annals of Mathematics Studies, No. 38, Princeton University Press, Princeton.

13. Motzkin, T. S., 1936, Beitrage zur Theorie der Linearen Ungleichungen, Inaugural Dissertation, Basel, 1933, Azriel, Jerusalem.

14. Penrose, R., 1955, A generalized inverse for matrices. *Proc. Camb. Philos. Soc.*, **51**, 3, 406–413.

15. Tucker, A. W., pp. 3–18 in [12].

16. Tucker, A. W., June 1962, Simplex Method and Theory, Notes on Linear Programming and Extensions, Part 62. The Rand Corporation, Santa Monica, Memorandum RM-3199-PR.

TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA.
AND
NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS